MATH 2030 3.00 – Elementary Probability Course Notes
Part II: Independence and Conditional Probabilities

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Independence and Conditional Probabilities

- Events A and B are said to be *independent* if

$$P(A \cap B) = P(A)P(B).$$

- If $P(B) \neq 0$ then the *conditional probability of A given B* is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Interpretation: If we repeat our experiment many times, one sees that $P(A \mid B)$ is the relative frequency with which A will occur, *out of just those repetitions in which B occurs*. So it is the likelihood that A will occur, if we're given the information that B will occur.

Likewise, if $P(B) \neq 0$ then independence $\Rightarrow P(A \mid B) = P(A)$. That is, knowing that B will occur doesn't affect the likelihood that A will occur. In other words, independence means that B occurring has no influence on A occuring.

Independence and Conditional Probabilities

Properties:

- Independent – $P(A \cap B) = P(A)P(B)$
  Mutually exclusive – $P(A \cup B) = P(A) + P(B)$.
  These are different; don't mix them up.
- $P(A \mid A) = 1$
- $P(A^c \mid A) = 0$
- $P(A^c \mid B) = 1 - P(A \mid B)$
  [pf: $P(B) = P(A \cap B) + P(A^c \cap B)$. Now divide by $P(B)$.]
- $P(B) = 0$ or $1 \Rightarrow B$ is independent of every event A.
- sampling with replacement $\Rightarrow$ independence.
  sampling without replacement $\Rightarrow$ dependence.

Probabilities $\longmapsto$ Conditional Prob's.

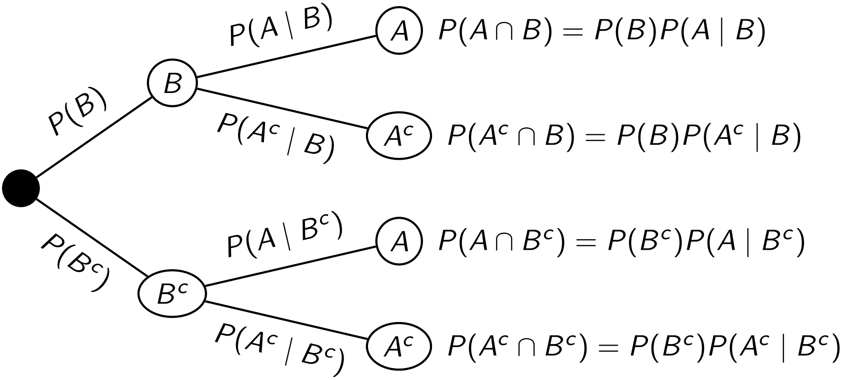
- **Eg:** Draw 2 cards in order, without replacement.
  $$P(\text{2nd is a } \heartsuit \mid \text{1st is a } \heartsuit) = \frac{P(\text{both are } \heartsuit)}{P(\text{1st is a } \heartsuit)} = \frac{\frac{13\times 12}{52\times 51}}{\frac{13\times 51}{52\times 51}} = \frac{12}{51}$$
  [which is what we'd have guessed anyway].
- **Roll 2 dice:** $P(\text{1st is a 3}) = \frac{1}{6} = P(\text{2nd is a 3})$, and $P(\text{both are 3's}) = \frac{1}{36} = \frac{1}{6} \times \frac{1}{6}$, so they are independent [as we'd have guessed]. Likewise
  $$P(\text{2nd is a 3} \mid \text{1st is a 3}) = \frac{P(\text{both are 3's})}{P(\text{2nd is a 3})} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}.$$
- **Eg:** Flip 4 coins. A is that all 4 flips agree.
  B is that flips 1 & 2 are H. C is that flip 1 is H.
  Then A and B are dependent, but A and C are independent.
  [Calculate: $P(A \mid C) = \frac{1}{8} = P(A) \neq P(A \mid B) = \frac{1}{4}$.
  The probability that the last 3 flips agree with the 1st doesn't depend on whether the 1st is H or T.]

Conditional Prob's $\longmapsto$ Probabilities.

- $P(A \cap B) = P(B)P(A \mid B)$
- $P(A \cap B \cap C) = P(C)P(B \mid C)P(A \mid B \cap C)$
- **Eg.** 2 urns. 1st has 3 red balls, 5 yellow balls. 2nd has 2 red and 3 yellow. Pick an urn at random and then a ball at random from that urn. What is the probability that it's red?

  A: ball is red. B: pick first urn. We know that $P(B) = P(B^c) = \frac{1}{2}$, $P(A \mid B) = \frac{3}{8}$, $P(A \mid B^c) = \frac{2}{5}$.
  So $P(A) = P(A \cap B) + P(A \cap B^c) = \frac{1}{2} \times \frac{3}{8} + \frac{1}{2} \times \frac{2}{5} = \frac{31}{80}$.

Tree Diagrams

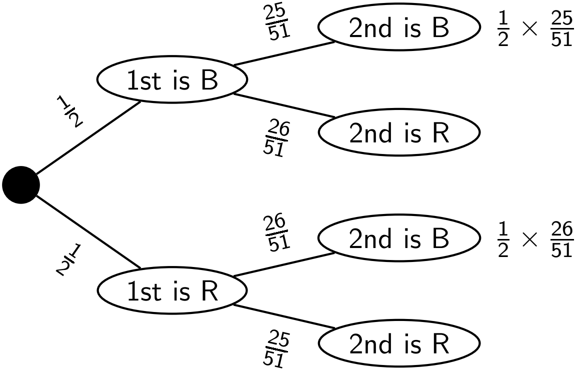


The probability for a node of the tree is the product of the conditional probabilities along the branches leading to that node.

Tree Diagrams

Eg: Draw 2 cards without replacement. $P(\text{2nd is black}) = ?$

1st method: By symmetry, the answer must $= \frac{1}{2}$.
2nd method: Go back to a model and count.
3rd method: Conditional probabilities:

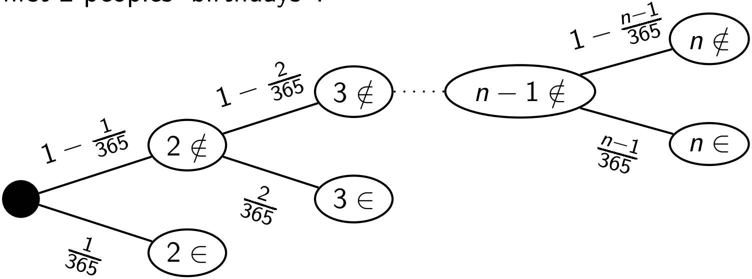


Adding the probabilities for these 2 nodes gives $\frac{1}{2}(\frac{25}{51} + \frac{26}{51}) = \frac{1}{2}$.

Birthday problem

Eg: n people. $P(\exists\ 2$ people with the same birthday$) = ?$

Put the people in some order and write eg. "$3 \notin$" as an abbreviation for "the 3rd person's birthday is different from the first 2 peoples' birthdays".



Then the probability of having shared birthdays is $1 - (1 - \frac{1}{365})(1 - \frac{2}{365}) \cdots (1 - \frac{n-1}{365})$. If $n = 30$ this is ≈ 0.7063, while if $n = 65$ it is ≈ 0.9977, and for $n = 80$ is ≈ 0.9999

Independence for 2 events

If A and B are indep., then the following are also indep.:

- A^c and B
  [pf: $P(A^c \cap B) = P(B) - P(A \cap B)$
  $= P(B) - P(A)P(B) = (1 - P(A))P(B) = P(A^c)P(B)$]
- A and B^c [likewise]
- A^c and B^c [likewise]

We'll see that this is the right way to generalize the notion of independence to 3 (or more) events.

Independence of 3 events

Three events A, B, C are said to be *independent* if
$P(A_1 \cap B_1 \cap C_1) = P(A_1)P(B_1)P(C_1)$
for $A_1 = A$ or A^c, $B_1 = B$ or B^c, and $C_1 = C$ or C^c.
In other words, if the following 8 conditions hold:

- $P(A \cap B \cap C) = P(A)P(B)P(C)$
- $P(A \cap B \cap C^c) = P(A)P(B)P(C^c)$
- $P(A \cap B^c \cap C) = P(A)P(B^c)P(C)$
- $P(A^c \cap B \cap C) = P(A^c)P(B)P(C)$
- $P(A \cap B^c \cap C^c) = P(A)P(B^c)P(C^c)$
- $P(A^c \cap B \cap C^c) = P(A^c)P(B)P(C^c)$
- $P(A^c \cap B^c \cap C) = P(A^c)P(B^c)P(C)$
- $P(A^c \cap B^c \cap C^c) = P(A^c)P(B^c)P(C^c)$

Independence of 3 events

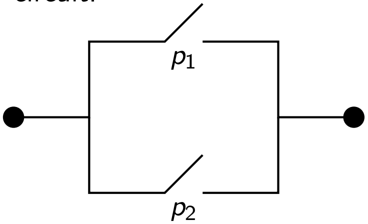
Consequences:

- A, B, C indep. $\Rightarrow$ pairwise independence.
  [eg. A and B are independent, because
  $P(A \cap B) = P(A \cap B \cap C) + P(A \cap B \cap C^c)$
  $= P(A)P(B)P(C) + P(A)P(B)P(C^c) = P(A)P(B)$]
- But A, B, C pairwise indep. $\not\Rightarrow$ A, B, C indep.
  [You'll work out a counterexample on an assignment]
- A, B, C indep. $\Rightarrow$ A indep. of any event got from B, C.
  [eg. $P(A \mid B \cap C) = P(A)$]
- Therefore all conditional probabilities in the tree diagram for A, B, C equal the corresponding probabilities.
- A, B, C indep. $\Leftrightarrow$
  A, B, C pairwise indep. and $P(A \cap B \cap C) = P(A)P(B)P(C)$.

Switches

Eg: 2 switches in parallel.

Switch 1 is closed (ie current flows through it), with probability p_1.
Switch 2 is closed, with probability p_2.
Assume the switches are open/closed independently of each other.
Problem: Find the probability that current flows through the circuit.

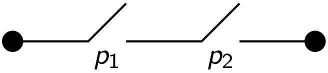


Solution: Write using $\cap$'s:

Let A_i be the event that switch i is closed. P(current flows)
$= P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) = p_1 + p_2 - p_1 p_2$.
Alternatively, $P(A_1 \cup A_2) = 1 - (P((A_1 \cup A_2)^c)$
$= 1 - P(A_1^c \cap A_2^c) = 1 - (1 - p_1)(1 - p_2) = p_1 + p_2 - p_1 p_2$.

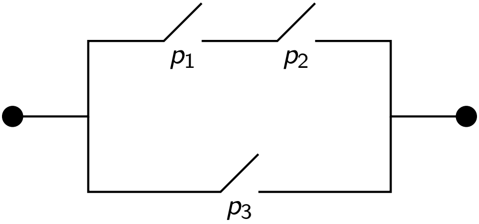
Switches

Eg: 2 switches in series.



A_i: event that switch i is closed.
$P(\text{current flows}) = P(A_1 \cap A_2) = p_1 p_2$

Eg: 3 switches.



3 switch example, cont'd

$A =$ event that current flows.
$A_i =$ event that switch i is closed.
$A_{12} = A_! \cap A_2$ is the event that current flows along the top.

- Inclusion/Exclusion: $P(A) = P(A_{12} \cup A_3)$
  $= P(A_{12}) + P(A_3) - P(A_{12} \cap A_3) = p_1 p_2 + p_3 - p_1 p_2 p_3$
- Complements: $P(A) = 1 - P(A^c) = 1 - P(A_{12}^c \cap A_3^c)$
  $= 1 - P(A_{12}^c)P(A_3^c) = 1 - (1 - p_1 p_2)(1 - p_3)$.
- Enumeration of alternatives:
  $A = A_3 \cup (A_{12} \cap A_3^c)$ and the latter are disjoint. So
  $P(A) = P(A_3) + P(A_{12} \cap A_3^c) = p_3 + p_1 p_2 (1 - p_3)$.

With each approach, the point is to write things in terms of intersections, so that independence applies.

Bayes rule: 2 alternatives

- Problem: Compute $P(B \mid A)$, knowing $P(B), P(A \mid B), P(A \mid B^c)$.
- Bayes rule: $\boxed{P(B \mid A) = \frac{P(B)P(A \mid B)}{P(B)P(A \mid B) + P(B^c)P(A \mid B^c)}}$
- Proof: $= \frac{P(A \cap B)}{P(A)} = \frac{P(A \cap B)}{P(A \cap B) + P(A \cap B^c)}$.
- **Eg:** 2 urns. Urn 1 has 3 Red & 5 Green balls. Urn 2 has 2 Red & 3 Green. Pick an urn at random and then a ball. If it's red, what are the chances we had picked the 1st urn? They will no longer be $\frac{1}{2}$, since the evidence favours urn 2, which has a higher % of reds.

  A: get a red ball. B: pick 1st urn. $P(B) = \frac{1}{2}$, $P(B^c) = \frac{1}{2}$, $P(A \mid B) = \frac{3}{8}$, $P(A \mid B^c) = \frac{2}{5}$.

  So $P(B \mid A) = \frac{\frac{1}{2} \times \frac{3}{8}}{\frac{1}{2} \times \frac{3}{8} + \frac{1}{2} \times \frac{2}{5}} = \frac{15}{31} < \frac{1}{2}$

Bayes rule: multiple alternatives

- Let $B_1, \ldots, B_n$ partition Ω. ie the B's are nonempty and disjoint, with $\cup B_i = \Omega$.
- Bayes rule:

$$\boxed{P(B_1 \mid A) = \frac{P(B_1)P(A \mid B_1)}{P(B_1)P(A \mid B_1) + \cdots + P(B_n)P(A \mid B_n)}}$$

- Proof: same as before.
  2-alternative case is $B_1 = B$, $B_2 = B^c$.
- **Eg: Medical screening test**
  A medical condition affects 1 person in 1,000. A test is 98% effective on healthy people and 99% effective on infected ones (ie it gives the "correct" answer that % of the time). If you test positive, what's the likelihood you have the condition?

Medical test, cont'd

- Define events A – test positive. B – are ill. B^c – are healthy.
- Problem asks for $P(B \mid A)$
- Information given is that $P(B) = 0.001$, $P(A^c \mid B^c) = 0.98$, $P(A \mid B) = 0.99$
- Therefore we compute $P(B^c) = 0.999$ and $P(A \mid B^c) = 0.02$ and apply Bayes.
- $P(B \mid A) = \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.02} \approx 0.0472$ – ie. rather small.
- Point is that Bayes $\Rightarrow$ false positives can swamp true ones, if the disease is rare. That is, such a screening test is only useful as a trigger for further tests. Doctors need to be able to explain this to patients.

Bayes rule cont'd

Other examples:

- Legal use of DNA testing:
  A – DNA match; B – guilty; B^c – innocent.
  Judge wants $P(B \mid A)$. Have same false positive issue as before: when a big database is tested, a match means much less than when an actual suspect is tested.
- Bayesian statistics:
  Suppose there is good evidence for a "prior probability" $P(B_k)$ for each alternative k. One then gathers data (ie observes an event A) and one revises the prior to get "posterior probabilities" $P(B_k \mid A)$. That is, likelihoods for the alternatives k, given the data.