Financial Economic Valuation of Guaranteed Minimum Withdrawal Benefits in Variable Annuities

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Abstract

We use No Arbitrage techniques to value an insurance feature available on many variable annuity (VA) policies called a Guaranteed Minimum Withdrawal Benefit (GMWB). The GMWB is portfolio insurance on an internal rate of return, as opposed to a guaranteed point-to-point investment rate of return. They are typically sold to retail investors, can not be easily traded in the secondary market and have penalties associated with early surrender. Like all VA policies – and in contrast to standard exchange traded options – insurance companies charge for this protection by deducting an ongoing *fraction of assets*, as opposed to an up-front fee.

Given all these non standard elements, we provide two extreme approaches to analyzing, valuing and managing the risk a GMWB. First, we take a *static actuarial* approach that assumes individual investors behave passively in utilizing the guarantee. In this case we show the product can be decomposed into a Quanto Asian Put plus a generic term-certain annuity. The opposite assumption is that investors are dynamically rational and seek to maximize the embedded option value by lapsing (a.k.a. surrendering or terminating) the product at an optimal time, i.e. once the expected present value of fees exceed the present value of benefits. We label this the *dynamic financial* approach, which leads to an optimal stopping problem akin to pricing an American put option, albeit complicated by the non-traditional payment structure.

Our main numerical result is that under a typical product specification which guarantees a 7% withdrawal, and assuming investment volatility of $\sigma = 20\%$, the theoretical cost of providing a GMWB ranges from 73 to 160 basis points of assets per annum, with the variation depending on the degree of what we label, *lapsation rationality.* In contrast to our estimates, recent GMWB products are only charging 30 to 50 basis points, even though the underlying annuity sub-accounts contain high-volatility investment choices. We suggest a number of reasons for the apparent under-pricing of this particular feature.

JEL Classification: G22

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1 INTRODUCTION AND MOTIVATION

Insurance companies in the U.S. have recently introduced a *new* type of financial guarantee into their already complex variable annuity (VA) policy line-up, called a Guaranteed Minimum Withdrawal Benefit (GMWB). VAs are very similar to mutual funds which bundle individual securities, such as stocks and bonds, into diversified units or trusts. Aside from their legal classification as an insurance policy as opposed to a registered security, VAs provide tax sheltered growth, but also embed a number of put-like derivative products that provide minimal guarantees on the account value at the earlier of maturity or death. See Milevsky and Posner (2001) or Brown and Poterba (2004) for a discussion of the features and reasons underlying the demand for variable annuities, and the possible relation to income taxes.

A recent innovation in this market, the GMWB, contains absolutely no life insurance component and is thus well within the domain of analysis of classical finance. The GMWB promises a minimal payout level from an initial investment capital – akin to a systematic withdrawal plan (SWiP) – for a fixed period of time, *regardless* of the performance of the underlying policy supporting the variable annuity. Typically, the policyholder might be guaranteed the ability to withdraw at least $7 per annum per $100 of initial investment until the original $100 have been fully exhausted. Thus, if the market performed poorly – and especially in the early years when the VA is purchased – the investor would be guaranteed a minimal *weighted average* return. Like all VA riders, and in contrast to standard exchange traded options, insurance companies charge for this downside protection by deducting an ongoing *fraction of assets*. These unique features differentiate the pricing of this derivative security from the standard Black-Scholes (1973) approach, where the option premiums are paid up-front and in advance.

The GMWB is not a trivial wrinkle in a small market. In fact, it is being offered by a growing number of insurance companies, although each with its own peculiar and distinguishing features[1]. The contribution of this paper is to (a) use No Arbitrage techniques to analyze insurance features in Variable Annuities, an area that has not received nearly as much academic attention as the mutual fund market, despite its $1 trillion size, and

[1]At last count, we found 12 different companies offering GMWBs on their Variable Annuities. They are: Lincoln National, Jackson National, Jefferson National, Pacific Life, Transamerica, ING Golden America, Manulife Financial, Sun Life of Canada, Hartford Life, American Skandia (via Prudential Financial), Travelers and USAllianz.

(b) provide two extreme approaches to analyzing, valuing and managing the risk a GMWB that are predicated on the degree of investor rationality in this market. Indeed, it is now well established that individual retail investors do not adhere to most the basic tenets of economic optimality. For example, Benartzi and Thaler (2001) document that investors in 401(k) pension plans use simple *1/n* heuristics to select mutual funds as opposed to using a mean-variance approach to diversify their portfolio. Other papers in the behavioral finance literature provide evidence that consumers can not be relied upon to optimally exercise financial options, such as executive and incentive options. This, of course, should impact the pricing of any (illiquid, non tradeable) derivative security offered to retail investors where a portion of the value is based on the counter-party optimally exercising the option. Our valuation algorithm is more than just a theoretical exercise to derive an abstract price for an illiquid instrument. The liability created by GMWBs should have a direct and measurable impact on the amount of capital (and reserves) insurance companies should be required to hold against these guarantees. Traditionally insurance companies have relied on the law of large numbers to set reserves which cover the risks $(1-\varepsilon)\%$ of the time. But as insurance companies venture into offering products which merge life insurance and financial (downside) protection, there is a need to value the *financial economic* risks they are undertaking, especially given the recent movement towards fair value accounting and risk-based capital in the insurance industry.

In this paper we present two extreme valuation algorithms – both within the framework of No Arbitrage pricing – for pricing the GMWB. First take a *static actuarial* approach that assumes individual investors behave passively in utilizing their guarantee. In this case we show how the rider can be decomposed into a Quanto Asian Put (QAP) plus a generic term-certain annuity. We believe this bifurcation has not been previously known in the literature and this obviously allows the insurance company to use QAPs to hedge the product. We rely on numerical techniques to price the embedded Asian options – see for example Turnbull and Wakeman (1991) or Milevsky and Posner (1998) for a review of the various approaches – and we provide numerical estimates for the value or hedging cost of the GMWB under this static actuarial approach.

The opposite assumption is that all investors buying these GMWB features are dynamically rational and seek to maximize the embedded option value by lapsing (a.k.a. surrendering or terminating) the product at an optimal time, i.e. once the expected present value of fees exceed the present value of benefits. We label this the *dynamic financial* approach, and its analysis leads to an optimal stopping problem akin to pricing an American put option, albeit complicated by the non-traditional payment structure. We formulate the optimal boundary as a linear complementarity problem and then use numerical PDE techniques to obtain pricing results.

Our main practical result is that under a stylized product specification which guarantees a 7% withdrawal, and assuming historical investment volatility of $\sigma = 20\%$, the cost of providing a GMWB ranges from 73 to 160 basis points of assets per annum, with the variation depending on the degree of what we label, *lapsation rationality.* Of course, our pricing does not allow for any profits, commissions, fees and transaction costs, akin to the celebrated Black-Scholes formula. Yet, in contrast to our estimates, we find the recent GMWB products that have been introduced in the market are only charging 30 to 50 basis points, even though the underlying annuity sub-accounts contain high-volatility investment choices. This conclusion is especially puzzling given the evidence provided in Milevsky and Posner (2001) that most return-of-premium riders on variable annuities are grossly over-priced.

During the last ten years there has been nothing short of an explosion of exotic options and financial guarantees that are being embedded within insurance policies. In fact, some have argued, for example Boyle (2003), that the options embedded within insurance policies are even more complex than those in standard OTC and exchange traded contracts. And, while the rationale for this phenomena requires some justification, the embedded options are at times quite challenging to price, value and hedge. Historically, they have been analyzed by a variety of academics and practitioners under the label of equity-linked policies, starting with the extension of Black-Scholes (1973) by Boyle and Schwartz (1977), and more recently Persson and Aase (1997), Gerber and Shiu (1999) as well as Milevsky and Posner (2001) and Windcliff, Forsyth and Vetzal (2001). In fact, we count more than 60 published papers – most of them from the insurance perspective – written on the topic within the last ten years alone. For a selected bibliography and recent book on the topic, we refer the interested reader to Hardy (2003). But, as mentioned earlier, the contribution of the paper is to take a financial economic approach to the (new) GMWB features that differentiates between various forms of rationality and contrast these values with actual pricing in the market.

The remainder of the paper is organized as follows. The next sub-section 1.1 provides a numerical example to explain the mechanics of a GMWB. In Section 2 we provide a stochastic modeling framework for the GMWB and discuss the real-world probability the feature will end-up "in the money". This metric will likely be used when computing a traditional insurance reserve for the guarantee. Section 3 provides the static actuarial analysis of the GMWB by decomposing the product into a term-certain annuity and a Quanto Asian Put. We also provide some numerical examples. Section 4 illustrates the dynamic financial perspective by solving the relevant optimal stopping problem and Section 5 concludes the paper.

1.1 Numerical Example of Product Specifics

Table #1 provides a numerical example of the payoff from a GMWB rider, assuming a particular sequence of quarterly investment returns for a typical variable annuity (VA) policy. The example assumes an initial investment of $100 and a guaranteed withdrawal of $7 per annum, which is $1.75 per quarter. At the end of each quarter an investment return is recorded and applied to the previous end-of-quarter's account balance. Thus, for example, after the first quarter of negative 12.24%, the balance in the VA policy was $87.76, then $1.75 was withdrawn. The next quarter resulted in a positive 10.06% return, and the account grew to $94.66, etc.

TABLE #1 Placed Here

Under this particular sequence, the *option payoff* starts at the end of the first quarter of the seventh year where the policy balance has fallen to a mere 0.17 dollars. Under a standard systematic withdrawal plan there is no longer enough to withdraw the requisite $1.75 per quarter and the policy is therefore 'ruined'. In fact, under this particular state of nature, the total amount withdrawn up to and including the first quarter of the seventh year is $45.42, due to the mostly poor performance of the investments during the first few years. The GMWB kicks-in and continues to provide an income of $1.75 until the entire $100 has been returned. Note that the entire $100 will be returned in exactly $100/1.75 = 57.14$ quarters which is $57.14/4 = 14.285$ years. At the end of 14.28 years the entire sum is returned and the guarantee *matures.* The insurance company backing the VA policy and the guarantee would be 'on the hook' for the remaining $100 - 45.22 = 54.78$ dollars, albeit paid over the remaining seven years.

Note that Table #1 represents but one of many millions of possible scenarios for the 14.28 years of the guarantee's life. Figures #1, #2 and #3 illustrate three other possible scenarios.

FIGURE #1, #2, #3 Placed Here

In the first one, the account is driven to a zero value at the beginning of the fifth year, and the insurance company pays the remaining \$1.75 per quarter until time 14.28. In the second case, the funds are exhausted in the middle of the eleventh year and the insurance company makes three years of additional payments. In the final case the variable annuity survives a 7% withdrawal for 14.28 years and the company is relieved of its obligations.

The option appears quite novel upon first inspection, since it starts paying-off at a random *ruin time* for the underlying investment-net-of-withdrawal process. A random maturity option was first analyzed by Carr (1998), but the product specifics are quite different since the random maturity (ruin time) in our product is determined endogenously and thus closer to a barrier option. We will discuss the precise nature of the embedded option in Section 3. The next section will set-up the model and examine the odds the GMWB feature will payoff.

2 MODELING FRAMEWORK

Let W_t denote the market value of the underlying VA at any future time $t \geq 0$, with an arbitrary (but innocuous) assumption that $w_0 = 100$ dollars. The most typical GMWB structure is that the policyholder is guaranteed to be able to withdraw *at least* $G = gw_0 = 7$ dollars per annum. The guarantee remains in effect until the entire \$100 has been disbursed, which at a minimum is a period of $100/7 = 14.28$ years. Thus, even in the extreme scenario where the initial $w_0 = 100$ collapses to a zero value one day after the policy is purchased, the investor will be made whole, albeit over an extended period of 14.28 years. Of course in any given year the policyholder is entitled to withdraw an amount less than $G = 7$ dollars, which would extend the life of the guarantee. Or, the policyholder could withdraw an amount greater than $G = 7$ dollars which would reduce both the value *and* life of the guarantee. These cases where the policyholder withdraws more or less than suggested by the guarantee – which falls under the category of dynamic strategies – will be carefully addressed in Section 4. In this section, we proceed by assuming the policyholder withdraws no more and no less than the $G = 7$ dollars per annum. This is called the *passive actuarial* approach.

Following most of the modern option pricing literature, we assume the actual dynamics of the VA policy – including withdrawals – obey the following stochastic differential equation (SDE). The value of the VA sub-account, gross of any withdrawals, obeys:

$$dS_t = (\mu - \alpha)S_t dt + \sigma S_t dB_t. \tag{1}$$

This is a standard modeling assumption that is used when pricing financial options. The symbol B_t denotes a standard Brownian motion with mean zero and variance t. The parameter (mu) μ is the real-world expected growth rate of the investment (net of withdrawals), the parameter α captures the insurance fee that pays for the guarantees, and finally (sigma) σ represents the volatility of the investment return.

The individual is assumed to invest $W_0 = w_0$ in a variable annuity, and so the dynamics of the actual VA policy is defined by:

$$dW_t = dS_t - \gamma_t dt, \tag{2}$$

where $0 \leq \gamma_t \leq W_t$ represents the withdrawals from the account, which can range from a low of zero, to as high as the actual account value W_t. In what follows, we assume that the withdrawal amount is exactly equal to the guaranteed amount $\gamma_t := G$, which is what we label the passive actuarial approach. In section 4 we remove the restriction and investigate the impact of assuming flexibility in the withdrawal rate. In the simple case,

$$dW_t = (\mu - \alpha)W_t dt - G dt + \sigma W_t dB_t, \qquad W_0 = w_0. \tag{3}$$

Using standard techniques which can be verified by Ito's lemma – see Karatzas and Shreve (1992) for details – the solution to the SDE in equation (3) can be written as:

$$W_T = e^{\left(\mu-\alpha-\frac{1}{2}\sigma^2\right)T+\sigma B_T} \left(w_0 - G \int_0^T e^{-\left(\mu-\alpha-\frac{1}{2}\sigma^2\right)t-\sigma B_t} dt \right) \tag{4}$$

The first thing to note about the dynamics in equation (3) and (4) is that since $G > 0$, which means that the process includes a forced dollar consumption, the value of W_t can in fact hit zero at some point $t > 0$. Although the exponential Brownian motion term is always positive, as soon as the integral term in equation (4) exceeds w_0/G, the quantity in the brackets will become negative. This is in contrast to a standard geometric Brownian motion, which is the term multiplying the brackets in equation (4) that can never hit zero in finite time. The guaranteed ability to withdraw G per annum until time $T = w_0/G$, is of value *if and only if* the process W_t hits zero prior to T. Indeed, for those sample-paths on which the ruin time occurs after T, the embedded put option has a zero payout since the minimum withdrawal would have been satisfied endogenously, *even* without an explicit guarantee provided by the insurance company.

Given the importance of the ruin time in the classification and understanding of this guarantee, we introduce the following notation for the probability of ruin of the process W_t, within the time period $[0, t]$. The function (xi) ξ_t is:

$$\xi_t = \Pr\left[\inf_{0 \leq s \leq t} W_s \leq 0\right] = \Pr\left[\int_0^t e^{-\left(\mu-\alpha-\frac{1}{2}\sigma^2\right)s+\sigma B_s} ds \geq \frac{w_0}{G}\right] := \Pr[\mathbf{X}_t \geq \frac{w_0}{G}], \tag{5}$$

where the new term $\mathbf{X}_t$ is defined equal to the integral in the middle of equation (5). The seemingly counter-intuitive relationship between the infimum of a process and the integral of an exponential Brownian motion, comes from the fact that equation (4) can only become negative once the integral $\mathbf{X}_t$ is greater than w_0/G. Note also the fact that $\mathbf{X}_t$ is monotonically increasing in t. Thus, once $\mathbf{X}_t$ exceeds w_0/G, which means that $W_t < 0$, it can never recover and go above zero.

It is quite easy to demonstrate that the probability of ruin ξ_t is increasing in the withdrawal rate G, and likewise, the greater the value of time t, the higher is the probability of ruin. In fact, although it is beyond the scope of the current analysis, one can actually obtain a precise analytic expression for ξ_t when $t \to \infty$. In the current context we are most interested in the value of ξ_T where $T = w_0/G$. In other words, we would like to know what the odds are that the investor would actually run out of money by the end of the guarantee period, assuming they withdrew the guaranteed amount.

2.1 The Real-World Probability of Ruin

Assume the arithmetic average return is expected (after management fees but prior to insurance fees) to be $\mu = 9\%$ per annum jointly with a historical market volatility of $\sigma = 18\%$. According to Morningstar statistics as reported by Milevsky and Posner (2001), the median sub-account volatility for the universe of variable annuity policies is 18%, with a 25th percentile of 16% and a 90th percentile of 25%. To be clear, these parametric assumptions imply that the *continuously compounded* rate of return (a.k.a. the instantaneous growth rate of the investment) is assumed normally distributed with a mean value of $0.09 - (0.18)^2/2 = 7.38\%$ and a standard deviation of 18%. Under a normality assumption, two thirds of the time the investment return will fall in a range of $7.38\% + 18\% = 25.38\%$ and $7.38\% - 18\% = -10.62\%$ which is consistent with broad-based U.S. indices. Also, we let the insurance fee for this particular rider alone be set to $\alpha = 0.40\%$ per annum, which is consistent with current market pricing of these products. In this case, the parameterized dynamics of the investment become:

$$dW_t = ((0.086)W_t - 7)\, dt + 0.18 W_t dB_t, \qquad w_0 = 100. \tag{6}$$

Using numerical PDE methods or moment matching approximations – described at greater length in Huang, Milevsky and Wang (2004) – to obtain the ruin probability during the first $T = 14.28$ years we find that $\xi_{14.28} = 11.7\%$. In other words, there is approximately an 88%

chance that *even* if the policy holder withdraws the maximum allowable amount each year, the policy will *survive* to the end of the guaranteed horizon.

However, if we increase the investment return volatility to $\sigma = 25\%$ per annum, the ruin probability increases to $\xi_{14.28} = 26.2\%$. And, if we reduce the expected (arithmetic average) return to $\mu = 6\%$ and maintain a high $\sigma = 25\%$ volatility, the probability of ruin increases to $\xi_{14.28} = 39.9\%$, which are clearly non-trivial amounts. Table #2 displays the probabilities under various risk and return combinations.

TABLE #2 Placed Here

Note that if the expected investment return is increased to $\mu = 12\%$ and the volatility of the return is set to 10%, the probability the withdrawals of $G = 7$ dollars per annum will actually exhaust (or ruin) the policy prior to time $T = 14.28$ is less than one half of a percent. The probability of *ex ante* usage range from 1% to 20% depending on the subjective asset return assumption and characteristics, these will impact the setting of traditional insurance reserves. The relevant question we are interested in is: *How much does it cost the insurance company to hedge this guarantee?* This will determine the financial economic value of the guarantee to the holder.

3 STATIC ACTUARIAL ANALYSIS

In this section we illustrate how to bifurcate the product into a collection of strip-bonds (or a term-certain annuity) and a complex option in the form of a Quanto Asian Put. Note that $g = G/w_0$ and by definition $T = 1/g$ (since the product terminates or matures when all the funds have been returned) and so we have that

$$W_T = w_0 e^{\left(\mu-\alpha-\frac{1}{2}\sigma^2\right)T+\sigma B_T}\left(1 - \frac{1}{T}\int_0^T e^{-\left(\mu-\alpha-\frac{1}{2}\sigma^2\right)s-\sigma B_s} ds\right). \tag{7}$$

The payoff of the *GMWB option* is:

$$\text{Option Payoff} := e^{-rT}\max[W_T, 0], \tag{8}$$

since the holder of the variable annuity policy is guaranteed to receive any remaining funds in the account at time $T = 1/g$. Remember that the policyholder is also entitled to the periodic income flow in addition to the (possibly zero) maturity value of the account. Thus, focusing on the future value of all cash-flows and payments, the maturity value of the *periodic income* is:

$$w_0 g \int_0^T e^{rt} dt = \frac{w_0 g}{r}\left(e^{rT} - 1\right) \quad (9)$$

The (No Arbitrage) time-zero present value of the GMWB cash-flow package is therefore:

$$e^{-rT} E_Q\left[\max[W_T, 0]\right] + \frac{w_0 g}{r}\left(1 - e^{-rT}\right). \quad (10)$$

where $E_Q[.]$ denotes the expectation under the Q-measure, under which the real-world drift μ is replaced by the risk-free rate r. We refer the interested reader to any standard textbook on derivative pricing to justify this substitution of measures.

Finally, for the GMWB to be fairly priced we must have at inception that the amount invested in the product w_0, is greater than or equal to the value of the cash-flow package, where $T = 1/g$.

$$w_0 \geq e^{-r/g} E_Q\left[\max[W_{1/g}, 0]\right] + \frac{w_0 g}{r}\left(1 - e^{-r/g}\right). \quad (11)$$

Equation (11) is one of our main results. It says that for the product to be fairly structured, the initial purchase price must equal the cost of the term-certain annuity plus the exotic option. For any given (r, σ) we can locate the (α, g) curve across which the product is fairly priced, which implies equality in equation (11).

We further claim that the option component is effectively a Quanto Asian Put (QAP) defined on an underlying security which is the inverse of the account price process. To illustrate this, define a new (reciprocal) process:

$$Y_t = S_t^{-1} = e^{-\left(r - \alpha - \frac{1}{2}\sigma^2\right)t - \sigma B_t}, \qquad Y_0 = 1, \quad (12)$$

One can think of Y_t as the number of VA sub-account units that one dollar can buy, akin to the number of Euros or Yen than one dollar can purchase in the currency market. The inverse, $S_t = Y_t^{-1}$, is the value of one VA sub-account unit in dollars, or the price of one Euro or Yen in USD. Now let:

$$\mathbf{A} := \frac{1}{T}\int_0^T Y_t dt, \qquad \mathbf{Y} := Y_T \quad (13)$$

The payoff from the GMWB option is:

$$\text{Option Payoff} := w_0 \frac{\max[1 - \mathbf{A}, 0]}{\mathbf{Y}} \quad (14)$$

This represents w_0 units of a Quanto (Fixed Strike) Asian Put option. In sum, scaling everything by the initial premium, a fairly priced product *at inception* implies the relationship:

$$e^{-r/g} E_Q\left[\frac{\max[1 - \mathbf{A}, 0]}{\mathbf{Y}}\right] + \frac{g}{r}\left(1 - e^{-r/g}\right) = 1 \quad (15)$$

Thus, our main qualitative insight is that under a static actuarial perspective, this product can be decomposed into the following items:

1. A term-certain annuity paying G per annum for a period of $T = w_0/G$ years, plus
2. A Quanto Asian Put (QAP) on the above-mentioned *reciprocal* variable annuity account.

For example, for an initial deposit of $w_0 = 100$ at a guarantee withdrawal rate of $G = 7$ per annum and an interest rate rate of $r = 0.06$, the time-zero cost of the term-certain annuity component is 67.15 dollars. The remaining 32.85 would go towards purchasing the option. One can think of a VA with a GMWB as consisting of 67% term-certain annuity and 32% Quanto Asian Put option. In contrast, at a (lower) interest rate of $r = 0.05$ the cost of the term-certain annuity would be (a higher) 71.46 dollars and only 28.54 would go towards purchasing the required option.

TABLE #3 Placed Here

Table #3 displays the required insurance fee that would lead to an equality in equation (11) under a number of different volatility values. We price the Asian option using a numerical technique which is described in the appendix. For example, if the VA guarantees a 7% withdrawal, and the pricing volatility is $\sigma = 20\%$, the fair insurance fee would be approximately 73 basis points of assets per annum. Stated differently, a financial package which offers a stream of \$7-per-annum income (in continuous time) plus a Quanto Asian Put that matures in exactly $T = 14.29$ years is worth exactly $w_0 = 100$, when the investment on which is the option is struck is 'leaking' a dividend yield of 73 basis points per annum. If the guarantee is reduced to $g = 4\%$ – which implies the product matures in $T = 25$ years – the fair insurance fee is only 23 basis points. Likewise, if the guarantee is increased to $g = 9\%$ – which implies the product matures in $T = 11.11$ years – the fair insurance fee is 117 basis points. As we mentioned in the introduction, the most common GMWB guarantee being offered on variable annuities is $g = 7\%$, which (even) under a conservative $\sigma = 15\%$ volatility implies an insurance fee of 40 basis points.

3.1 Lapsation and Mortality

The possibility of irrational policyholder lapsation and early death will only serve to reduce the value and hedging cost of the guarantee. Indeed, if we assume the guarantee will be terminated upon death of the policyholder who is currently aged x – and the beneficiary of the VA only receives the market value at the time of death – then the term-certain annuity must be replaced with a life annuity that terminates at $T = w_0/G$. In this case, the time zero cost becomes:

$$\text{Cost of Term-Certain Annuity} = \int_0^T G({}_tp_x)e^{-rt}dt \leq \int_0^T Ge^{-rt}dt, \tag{16}$$

and thus, mortality will reduce the marginal cost of providing the guaranteed GMWB. A popular analytic representation for $({}_tp_x)$ that can be used to price equation (16) is the Gompertz-Makeham law of mortality. See Carriere (1994) for details under which the conditional survival probability is:

$$({}_tp_x) = e^{\left(b\lambda_x(1-e^{t/b})\right)}, \qquad \lambda_x = \lambda + \frac{1}{b}e^{(x-m)/b}, \tag{17}$$

where b is a scale parameter and λ_x is the instantaneous hazard rate and the implicit λ and m are pre-specified constants. And, if we further assume a *constant* annual lapsation rate of h, which implies that only a fraction $\exp\{-hT\}$ will actually hold the VA to the end of the $T = w_0/G$ years. In general, if we let T denote the earlier rate of death and lapsation, the capital market (i.e. No Arbitrage) time-zero value of product under a *static actuarial* approach is:

$$e^{-r\mathbf{T}}E_Q\left[W_{\mathbf{T}}, 0\right] + \int_0^{\mathbf{T}} Ge^{-rt}dt, \tag{18}$$

where the random variable (stopping time) $\mathbf{T}$ is the earlier of death, lapsation or $1/g$. In sum, although Table #3 provides a value (or hedging cost) for the GMWB under the assumption that everyone 'behaves' exactly as predicted, in reality the insurance company can push the pricing to even lower levels by assuming a fairly high (real world) probability that the random variable $\mathbf{T} < 1/g$. This fact might explain the reason why observed GMWB fees in practice appear lower than dictated by Table #3. Of course, an alternative reason for apparent underpricing is that the base insurance fee on the VA product without the rider is enough to subsidize the extra cost of the relatively more expensive GMWB. Of course, all of this is predicated on the *static actuarial* approach that policyholders do not deviate from the $\gamma_t = G$ dollars per annum withdrawal. But, in fact, it might make sense for the policyholder to withdraw more or less than the minimum, even if it reduces the base of the guarantee, if the account has performed sufficiently well, making the original guarantee less valuable. It is not clear *a priori* the conditions under which this would make sense. This bring us to the next section which covers the pricing of these guarantees in perfect and complete financial markets where all counter-parties are fully rational and 'option value' maximizers.

4 DYNAMIC FINANCIAL HEDGING

In this section we employ classical *American option* pricing techniques to obtain a dynamic model of the GMWB, assuming policyholders are fully rational and lapse (i.e. withdraw more or less from) the product when it is to their economic advantage. As we argued in the introduction to the paper, the true 'cost' of the embedded guarantee lies somewhere between the *static actuarial* embedded option cost and the *dynamic financial* hedging cost.

It is important to note that once we include strategic lapsation as an option for the policyholder, the contingent deferred surrender charge (DSC) becomes an important factor in driving optimal behavior. Recall that most variable annuities impose a penalty if the product is lapsed or surrendered prior to maturity. This penalty is calculated as a fraction of the account value at the time of surrender, and can range from 10% to 0% depending on the product, company, and the time that has elapsed since the policy was acquired. And, while current practice in the industry is that the DSC goes exclusively towards paying commissions and brokerage fees – and is not used for risk management or hedging purposes – this penalty does induce the policyholders to continue holding the product and paying the ongoing asset-based management fees, even though the embedded option is far out-of-the-money. From a dynamic point of view, we must therefore work with a DSC curve or schedule in any optimal stopping model that attempts to capture the salient features of the product. We refer the interested reader to Milevsky and Salisbury (2001) for an in-depth analysis of the interaction between contingent deferred surrender charges (DSC) and the proper hedging of variable annuity secondary guarantees that are paid with an asset-based fee. In this paper we simply use a parametric version of this DSC and optimize around this curve.

4.1 The American Put Option

We begin by reviewing the required notation and background from the theory of American option pricing which we use in our model. We let V_t denote the value of a contingent claim, which depends on the traded underlying security price S_t.

$$
\begin{aligned}
V_t &\geq f(S_t), \quad \forall t \\
V_\tau &= f(S_\tau).
\end{aligned}
\tag{19}
$$

In a complete market – where the proper premium has been charged – the seller of protection is hedged against the worst-case scenario. The security price process obeys:

$$dS_t = rS_t\,dt + \sigma S_t\,d\tilde{B}_t, \tag{20}$$

where $\tilde{B}_t = B_t + t(\mu - r)/\sigma$. Define a new probability measure Q by re-weighting the probabilities so that $\tilde{B}_t$ is a Brownian motion under Q , which is the risk neutral probability measure. It follows by Itô's formula that $e^{-rt}S_t$ is a martingale under Q. This implies that the discounted value $e^{-rt}V_t$ of any self-financing portfolio is as well. Equivalently

$$dV_t = rV_t\,dt + dM_t \tag{21}$$

where M_t is a Q-martingale. To price, we look for an M_t such that V_t dominates $f(S_t)$ for every t. The cash flow from the hedge, if exercised at a stopping time η is $f(S_t)\,dR^\eta_t$, where

$$R^\eta_t = \begin{cases} 1, & t < \eta \\ 0, & t \geq \eta \end{cases} \tag{22}$$

The hedge, incorporating the payout, satisfies

$$dV_t = rV_t\,dt + dM_t + f(S_t)dR_t. \tag{23}$$

With optimal exercise (ie $\eta = \tau$), M_t will be a Q-martingale. In general – even with suboptimal exercise – it will be a Q-supermartingale. On the other hand, the value of hedge should be a function of the stock price, dropping to zero after exercise:

$$V_t = v(t, S_t)R_t. \tag{24}$$

Substituting into Itô's formula gives:

$$dV_t = \left[v_t + rS_t v_x + \frac{\sigma^2 S_t^2}{2} v_{xx}\right] dt + v dR_t + \sigma S_t v_x\,d\tilde{B}_t. \tag{25}$$

Equating the two expressions,

$$\left[v_t + rS_t v_x + \frac{\sigma^2 S_t^2}{2} v_{xx} - rv\right] dt + \left[v - f(S_t)\right] dR_t = dM_t - \sigma S_t v_x\,d\tilde{B}_t. \tag{26}$$

The RHS is a supermartingale in general, and a martingale under optimal exercise. Since the LHS is of bounded variation, it must be ≤ 0 in general, and $= 0$ under optimal exercise. That is,

$$v_t + rxv_x + \frac{\sigma^2 x^2}{2} v_{xx} - rv \leq 0 f - v \leq 0 \tag{27}$$

with at least one holding with equality. This is a linear complementarity problem – a.k.a. free boundary value problem – whose solution can be found numerically to give the option price. With this background, we now return to the pricing of the GMWB.

4.2 Hedging the GMWB

Let W_t be the value of the variable annuity account with a guaranteed minimum withdrawal benefit (GMWB) rider. Associated with this extra guarantee is the insurance fee $\alpha\%$ and the contingent deferred surrender charge (CDSC) of $\kappa\%$. As in the *static* case, these fees are imposed solely for hedging purposes. The GMWB allows up to Gdt to be withdrawn in time dt, so the CDSC applies only to withdrawals in excess of Gdt. The nominal withdrawal rate G is set as being a fixed percentage g, of the initial account value w_0. The terms of the GMWB contract specify that as long as the rate of withdrawals stays below G, the account holder may eventually accumulate withdrawals of A_t from the account, even if doing so would ordinarily drive the account to zero. Initially the guarantee level equals the account value, $A_0 = x_0 (= W_0)$. But W_t then fluctuates. Withdrawals decrease both W_t and A_t. If withdrawals ever occur at a rate higher than G, then not only is the CDSC imposed, but after the guarantee level and account value are debited by the withdrawal, the guarantee level is reset to the smaller of its value and the account value. As argued in the introduction, these provisions are idealized versions of ones from several existing variable annuities. Other terms could be analyzed the same way, for example with a CDSC that varies with time, or with provisions for resetting the withdrawal rate G.

Mathematically, we use the same GBM model (as in the static section) for W_t:

$$dW_t = (r - \alpha) W_t \, dt + \sigma W_t \, d\tilde{B}_t - \gamma_t \, dt \tag{28}$$

under the Q-measure. Here γ_t models continuous withdrawals from the account, which may or may not equal the allowed amount G. Similarly

$$dA_t = -\gamma_t \, dt \quad \text{provided } \gamma_t \leq G = g x_0, \tag{29}$$

but if $\gamma_t > G$ dollars, then A_t jumps to $\min(A_t, W_t)$. There are similar expressions in the case of lump-sum withdrawals $\Delta\gamma > 0$, but for simplicity sake we will carry out the analysis in the continuous case only. We wish to hedge this account. Write $V_t = v(A_t, W_t)$ for the value of the hedge. Insurance and DSC fees are retained in the hedge, so

$$dV_t = rV_t \, dt + dM_t - f(\gamma_t) \, dt, \quad \text{where}$$

$$f(\gamma_t) = \begin{cases} \gamma_t, & \text{if } \gamma_t \leq G \\ G + (1-\kappa)(\gamma_t - G), & \text{if } \gamma_t > G \end{cases} \tag{30}$$

An analysis similar to that of the American option lets us solve numerically for the hedging cost $v(a, w)$ by solving a free boundary value problem numerically. In contrast to the classical American put, there is no longer an initial fee on which to base the hedge – the initial value of the hedge is constrained to equal the initial account value, so the hedge must be financed through the insurance fee and DSC. In fact, our real problem is to determine, for a given value of κ of the DSC, what is the α that allows the guarantee to be hedged. In mathematical terms, we carry out an interactive solution and locate a *fixed point* in terms of α. For a given α we solve the free boundary value problem to give an initial hedging cost $v_\alpha(w_0, w_0)$. We then adjust α, resolve the PDE, readjust α, etc., to converge on the value of α that makes

$$v_\alpha(w_0, w_0) = w_0. \tag{31}$$

In principle, this might give an α that depends on the initial investment. But recalling that G is a linear function of w_0, there is, in fact, a scale invariance in the problem from which it can be shown that the same α works for all levels of w_0.

It turns out that the optimal withdrawal strategy γ_t amounts to withdrawing at an arbitrary large rate when W_t lies above some value $L(A_t)$, and to withdraw at the contracted rate G when $W_t \leq L(A_t)$. It should be emphasized that this *optimal* withdrawal strategy is not necessarily optimal from the point of maximizing the investor's expected utility - rather it is optimal in inflicting harm on the issuer of the policy. It is the worst-case scenario from the point of view of the hedger. Again, the capital-markets cost is the price of eliminating all possibility of shortfall. If one is willing to accept some positive shortfall probability or to make modelling assumptions about sub-optimal withdrawal behaviors, the hedging cost can be reduced.

Applying Itô's lemma to V_t, as we did for the American option,

$$dV_t = rV_t\,dt + dM_t - f(\gamma_t)\,dt \tag{32}$$

$$\begin{aligned} dv(A_t, W_t) &= v_a\,dA_t + v_w\,dW_t + \frac{1}{2}v_{ww}\,d\langle W\rangle_t \\ &= -v_a\gamma_t\,dt + (r-\alpha)W_t v_w\,dt + v_w\sigma W_t\,d\tilde{B}_t - v_w\gamma_t\,dt + \frac{\sigma^2 W_t^2}{2}v_{ww}\,dt. \end{aligned} \tag{33}$$

Equating gives

$$\Big[(r-\alpha)W_t v_w + \frac{\sigma^2 W_t^2}{2}v_{ww} - rv\Big]\,dt + \Big[f(\gamma_t) - \gamma_t v_w - \gamma_t v_a\Big]\,dt = dM_t - v_w\sigma X_t\,d\tilde{B}_t \tag{34}$$

where the RHS is a supermartingale, and a martingale under the optimal choice of γ. Thus as before,

$$\left[(r-\alpha)wv_w + \frac{1}{2}\sigma^2 w^2 v_{ww} - rv\right] + \left[f(\gamma) - \gamma v_w - \gamma v_a\right] \le 0 \quad \text{for every } \gamma, \tag{35}$$

with equality for some γ. Because f is piecewise-linear, there are three critical cases, namely $\gamma = 0$, $\gamma = G$, and $\gamma = \infty$. We arrive at the free boundary value problem

$$(r-\alpha)wv_w + \frac{\sigma^2 w^2}{2} v_{ww} - rv \le 0 \tag{36}$$

$$(r-\alpha)wv_w + \frac{\sigma^2 w^2}{2} v_{ww} - rv + G[1 - v_w - v_a] \le 0 \tag{37}$$

$$(1-\kappa) - v_w - v_a \le 0, \tag{38}$$

with equality in at least one case. Note in particular that the guarantee level α plays the same role in the second equation as time t did in the American put option problem. The numerical techniques used to solve the two problems are virtually identical.

4.3 Numerical Comparison of Static versus Dynamic

Table #4 provides some comparisons between the static actuarial and dynamic financial 'fair value' assuming the contingent deferred surrender charge of $k = 1\%$.

Table #4 Placed Here

For example, under a $g = 7\%$ withdrawal rate and a pricing volatility of $\sigma = 20\%$, the numerical solution to the system of PDEs in equations (36) to (38) leads to an insurance fee of 160 basis points of assets per annum. This can be compared to the 73 basis points required under the static actuarial case. When volatility is increased to $\sigma = 30\%$, the required insurance fee jumps to 565 basis points of assets. We remind the reader that these numbers are derived under the assumption that $\kappa = 1\%$ and that the insurance company can recover a portion of the hedging cost when the product is lapsed or surrendered by imposing the 1% penalty. In practice, if the risk management division within the insurance company can not 'use' or gain access to the 1% fee, the required α insurance fee would be even higher.

5 CONCLUSION

In this paper we develop two extreme approaches to analyzing a novel type of derivative security, called a Guaranteed Minimum Withdrawal Benefit (GMWB), which is an insurance rider offered on Variable Annuity (VA) policies. VA policies are a close cousin to mutual funds, but offer additional peformance-based guarantees. First, we take a *static actuarial* approach that assumes individual investors behave passively in utilizing the embedded guarantee. In this case, we show how the product can be bifurcated into a type of Quanto Asian Put (QAP) plus a generic term-certain annuity. At the other extreme, and in contrast to a static actuarial approach, we assume that investors are fully rational and seek to maximize the embedded option value by lapsing (a.k.a. surrendering or terminating) the product at the optimal time, i.e. once the expected present value of fees exceed the present value of benefits. We label this the *dynamic financial* approach, which leads to an optimal stopping problem akin to pricing an American put option, albeit complicated by the non-traditional payment structure. Our contribution lies in (i) bifurcating the product into its respective (simpler) derivative components, (ii) managing the non-traditional payment scheme, which is ia basis points of assets versus up-front payments, and (iii) introducing the distinction between two extreme valuation approaches, and (iv) discussing the optimal policy in this case. In other words, the less 'rational' the insurance company believes its target market is, the less they have to charge for the guarantee. This is quite different from pricing an America option freely traded in the open market, where the seller of protection can not afford the luxury of assuming less than full rationality on the part of the buyer.

In fact, the VA market in the U.S. – which is a $1 trillion dollar industry according to Morningstar Inc. – has not had anywhere near the intense level of financial economic analysis and scrutiny compared to the mutual fund industry. And while some might argue that the analysis of VAs should be left to the insurance literature, we disagree, since many of the issues raise subtle foundations of derivative pricing. In fact, this market provides a robust laboratory for testing theories of incomplete markets and frictions, given the restricted nature of these products.

On a practical side, our numerical results indicate that the current practice of charging between 30 to 50 basis points of assets on an ongoing basis for a typical 7% GMWB is not sufficient to cover the capital market hedging (No Arbitrage) cost of the guarantee, assuming a 20% pricing volatility. This is regardless of whether we take a static actuarial or dynamic financial approach to the problem. In fact, given the long-dated nature of the embedded options, it is quite likely that the *pricing* (implied) volatility would be even higher, if the company chose to use the capital markets to offset its risk. This under-pricing result stands in contrast to earlier work by Milevsky and Posner (2001) in which they show that the standard rider available on VA policies – the return of premium guarantee – is worth less than 5 to basis points of assets, compared to the observed fee of more than 100 basis points.

We offer a number of justifications at to why insurance companies might be pricing the GMWB rider at lower than what we perceive to be the fair *capital markets* cost.

1. The insurance company assumes a high level of irrational lapsation and possible mortality that would somehow further reduce the required fee, as we described in section 3.1

2. The GMWB rider/feature is being subsidized by the standard insurance fee. This is consistent with the over-pricing of standard features in VA policies.

3. The company uses a reserving methodology to manage the risk that differs from the capital market perspective. In other words, the do not hedge the risk using options, but simply compute a premium based on a real-world probability of less, as per our discussion is section 2.1

Of course, neither of these three explanation will protect the company in the event a secondary market develops for these products and consumer rationality increases to the point where they exercise their options at the optimal time. In sum, future research will continue to examine an appropriate and realistic hedging strategy for GMWB – and other more recent innovations on the border of life insurance and financial markets – in the presence of the usual collection of market imperfections.

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6 TECHNICAL APPENDIX

Pricing the Quanto and Computing the Ruin Probability.

Our model requires a quick and robust method of computing two important quantities for which there are no analytic solutions. They are:

$$A(w,t) := E[W_t | W_t > 0, W_0 = w], \quad P(w,t) := \Pr[W_t \le 0 | W_0 = w], \tag{39}$$

when W_t obeys the following stochastic differential equation (SDE):

$$dW_t = (\mu W_t - 1)\, dt + \sigma W_t \, dB_t, \qquad W_0 = w, \tag{40}$$

where μ, σ are the drift and diffusion coefficients and B_t is the Brownian motion driving the process. We refer the interested reader to the paper by Huang, Milevsky and Wang (2004) for extensive details and a comparison of alternative methods. Here we simply provide a high-level sketch of the algorithm. To start, $P(w,t)$ in equation (39) satisfies the Kolmogorov backward equation.

$$P_t + (\mu w - 1)\, P_w + \frac{1}{2}\sigma^2 w^2 P_{ww} = 0 \tag{41}$$

with a terminal condition

$$P(w_T, T) = 1 - H(w_T - y) \tag{42}$$

where $H(w)$ is the Heaviside function and w_T is the wealth at T. This is a second order linear partial differential equation which we can numerically solve by a θ-method which can be written as follows:

$$\frac{P_j^{(n+1)} - P_j^{(n)}}{\delta t} + (\mu w_j - 1)\left(\theta \frac{P_{j^*}^{(n+1)} - P_{j^*-1}^{(n+1)}}{\delta w} + (1-\theta)\frac{P_{j^*}^{(n)} - P_{j*-1}^{(n)}}{\delta w}\right)$$
$$+ \frac{\sigma^2 w_j^2}{2}\left(\theta \frac{P_{j+1}^{(n+1)} + P_{j-1}^{(n+1)} - 2P_j^{(n+1)}}{\delta w^2} + (1-\theta)\frac{P_{j+1}^{(n)} + P_{j-1}^{(n)} - 2P_j^{(n)}}{\delta w^2}\right) = 0, \tag{43}$$

where $P_j^{(n)}$ is a grid function which approximates $P(w,t)$ on the grid points $(w_j,\ t_n)$. A uniform grid with equal spacing δt and δx is used in our algorithm. The parameter θ can be arbitrarily selected, but when $\theta = 1/2$ it corresponds to the well-known second order Crank-Nickolson scheme. An upwind scheme is used for the first order derivative P_w, where the variable j^* is either j or $j+1$, depending on the sign of the coefficient. For any implicit method where $0 < \theta \le 1$, numerical boundary conditions must be provided on the computational boundaries $j = 0$ and $j = J$. This can be derived as:

$$P_0^n = 1, \qquad j = 0 \qquad \text{and} \qquad P_J^n = 0, \qquad j = J. \tag{44}$$

The case $j = 0$ and $j = J$ correspond to the $w_0 = 0$ and $w_J = W$ which are the boundaries of the truncated computation domain for calculating the probability W_t is less than some value at a *fixed* time. Likewise, for calculating the probability W_t is less than some value at *any* time, we use $j = 0$ and $j = J$ with respect to the $w_0 = y$ and $w_J = W$. These are the boundaries of the truncated computation domain. The terminal condition is:

$$P_j^N = 1 - H(w_j - y). \tag{45}$$

With these boundary conditions and the terminal conditions the discrete equations can be solved by matching from time t_n to t_{n+1}, starting from $n = 0$. At t_{n+1}, the equations for $P_j^{(n+1)}$ can be arranged from equation (43). In this space, we can solve for all the probabilities by iteration. For the expected value, which is $A(w, t)$ in equation (39), we can apply the same method.

TABLE #1: Numerical Example				
Time	**Investment**	**Balance**	**SWiP**	**GMWB**
Period	**Return**	**E.O.Q**		
0.25	-12.24%	$87.76	$1.75	$1.75
0.5	10.06%	$94.66	$1.75	$1.75
0.75	-6.43%	$86.94	$1.75	$1.75
1	-26.40%	$62.70	$1.75	$1.75
1.25	-17.30%	$50.41	$1.75	$1.75
1.5	-14.22%	$41.74	$1.75	$1.75
1.75	-7.05%	$37.17	$1.75	$1.75
2	-7.42%	$32.79	$1.75	$1.75
2.25	6.95%	$33.19	$1.75	$1.75
2.5	5.76%	$33.25	$1.75	$1.75
2.75	5.79%	$33.33	$1.75	$1.75
3	-1.68%	$31.05	$1.75	$1.75
3.25	3.13%	$28.38	$1.75	$1.75
3.5	16.57%	$22.22	$1.75	$1.75
3.75	-15.73%	$17.25	$1.75	$1.75
4	3.47%	$16.04	$1.75	$1.75
4.25	14.40%	$16.35	$1.75	$1.75
4.5	14.02%	$16.64	$1.75	$1.75
4.75	4.56%	$15.57	$1.75	$1.75
5	8.67%	$15.02	$1.75	$1.75
5.25	-9.40%	$12.02	$1.75	$1.75
5.5	0.70%	$10.34	$1.75	$1.75
5.75	-4.59%	$8.20	$1.75	$1.75

TABLE 1 continued.

Time	**Investment**	**Balance**	**SWiP**	**GMWB**
Period	**Return**	**E.O.Q**		
6	5.77%	$6.82	$1.75	$1.75
6.25	1.75%	$5.16	$1.75	$1.75
6.50	0.05%	$3.41	$1.75	$1.75
6.75	16.15%	$1.93	$1.75	$1.75
7	-6.83%	$0.17	$0.17	$1.75
7.25	14.98%	-	-	$1.75
7.50	0.33%	-	-	$1.75
7.75	-4.33%	-	-	$1.75
8	-1.22%	-	-	$1.75
8.25	-3.88%	-	-	$1.75
8.50	23.84%	-	-	$1.75
8.75	3.70%	-	-	$1.75
9	2.36%	-	-	$1.75
9.25	-18.28%	-	-	$1.75
9.50	3.68%	-	-	$1.75
9.75	8.31%	-	-	$1.75
10	-1.54%	-	-	$1.75
10.25	18.62%	-	-	$1.75
10.5	-15.57%	-	-	$1.75
10.75	-9.92	-	-	$1.75
11	10.66	-	-	$1.75
11.25	-1.58	-	-	$1.75
11.5	-17.23	-	-	$1.75
11.75	-1.05	-	-	$1.75
12	13.94	-	-	$1.75
12.25	-5.09	-	-	$1.75

Table 1 continued

Time	**Investment**	**Balance**	**SWiP**	**GMWB**
Period	**Return**	**E.O.Q**		
12.5	1.51%	-	-	$1.75
12.75	-4.33%	-	-	$1.75
13	12.06%	-	-	$1.75
13.25	19.85%	-	-	$1.75
13.50	4.76%	-	-	$1.75
13.75	-13.18%	-	-	$1.75
14	5.43%	-	-	$1.75
14.25	-9.83%	-	-	$1.75

Hypothetical return, cashflow and end-of-quarter account balance comparing a regular systematic withdrawal plan (SWiP) against the payoff from a Guaranteed Minimum Withdrawal Benefit (GMWB).

TABLE #2					
SWiP Probability of Ruin within 14.28 years					
40bp Insurance fee with net investment return (μ) and volatility (σ)					
	$\mu = 4\%$	$\mu = 6\%$	$\mu = 8\%$	$\mu = 10\%$	$\mu = 12\%$
$\sigma = 10\%$	19.0%	7.0%	1.7%	0.3%	0.04%
$\sigma = 15\%$	31.4%	18.5%	9.3%	4.1%	1.6%
$\sigma = 18\%$	37.8%	25.5%	15.5%	8.6%	4.4%
$\sigma = 25\%$	49.9%	39.6%	30.5%	22.2%	15.5%

In absence of an explicit GMWB, the process of withdrawing \$7 (per annum) for each \$100 of original principal – which is sometime called a Systematic Withdrawal Plan – might drive the portfolio to 'ruin' within 14.28 year, the time over which the \$7 would be recuperated. The above table computes the probability this event would occur under a variety of (real world) drift and volatility assumptions. See appendix for algorithm used to compute ruin probabilities.

TABLE #3: The Impact of the GMWB Rate and the Volatility of the Sub-account on the Required Fee			
Guarantee	Maturity (yrs)	Investment Volatility	
Rate	T = 1/g	σ = 20%	σ= 30%
4%	25.00	23	60
5%	20.00	37	90
6%	16.67	54	123
7%	14.29	73	158
8%	12.50	94	194
9%	11.11	117	232
10%	10.00	140	271
15%	6.67	272	475

Table assumes a 5% pricing interest rate. The table displays the required insurance fee to hedge the GMWB assuming everyone holds the product to maturity (i.e. the static actuarial analysis). The maturity of the Quanto Asian Put (QAP) is the inverse of the guarantee withdrawal rate since that is the time at which the original principal has been recovered

TABLE #4: Static (actuarial) vs. Dynamic (financial)		
Volatility	Static	Dynamic (DSC = 1%)
σ	$\alpha*$	$\alpha*$
15%	40	*tk*
18%	59	*tk*
20%	73	160
25%	113	320
30%	158	565

Assumes: 7% guaranteed withdrawal rate and 5% (pricing) interest rate. The expected (Q-measure) value of the account at maturity is 58.29% of the initial investment. The discounted expected (i.e. option) value is 28.53% of the initial investment. Under the static actuarial case, 28.53% of premium is used to purchase the Quanto Asian Put, while the remaining 71.47% should go towards buying a term-certain annuity paying $7-per-$100 for a period of 14.28 years. The purpose of this comparison is to note the higher required fees if we assume the individual will lapse the product if the option is no longer worth paying for (dynamic) versus if the investor holds the product to maturity (static).

Figure #1: Example of Policy Value under 7% withdrawal and Investment Returns

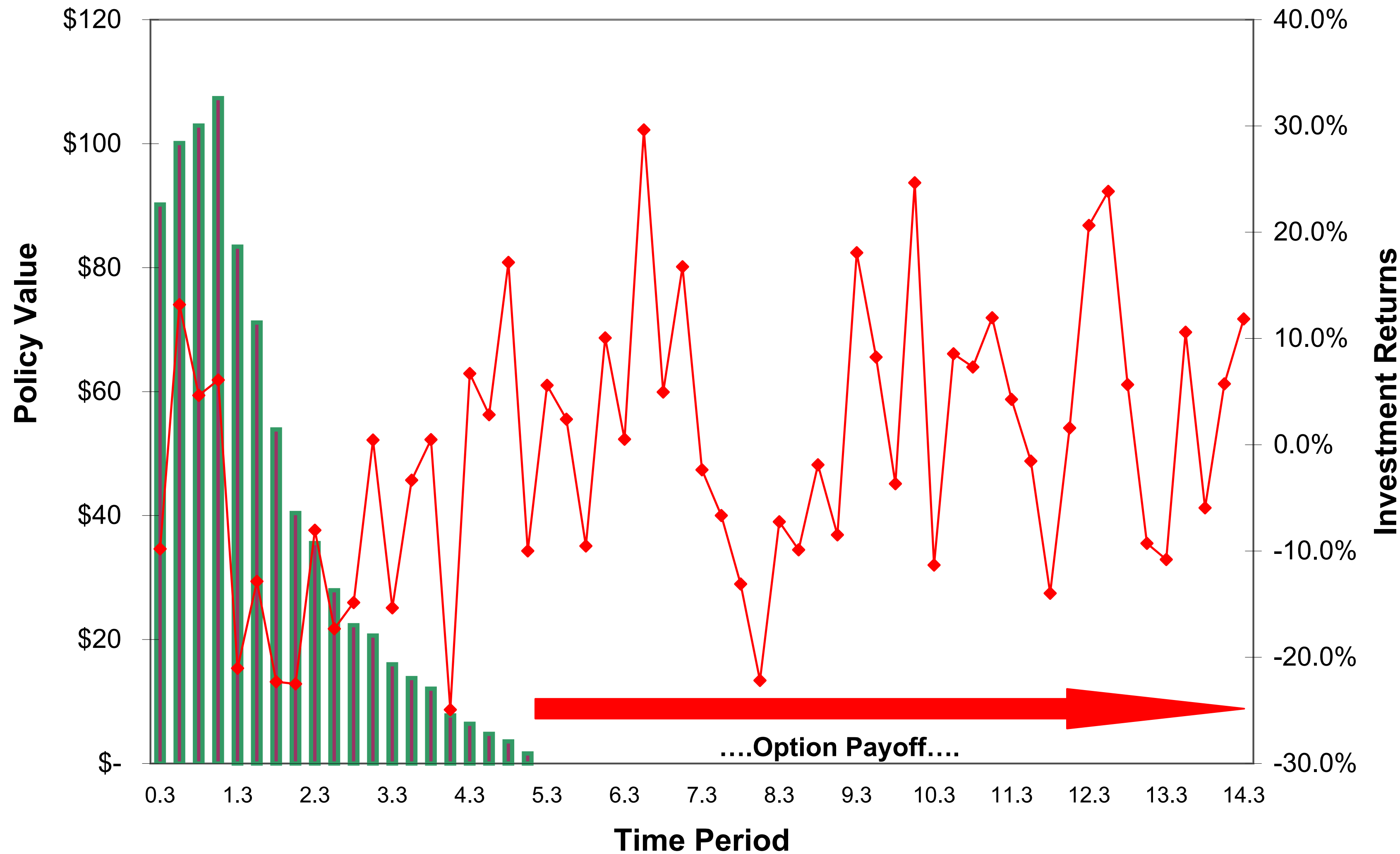
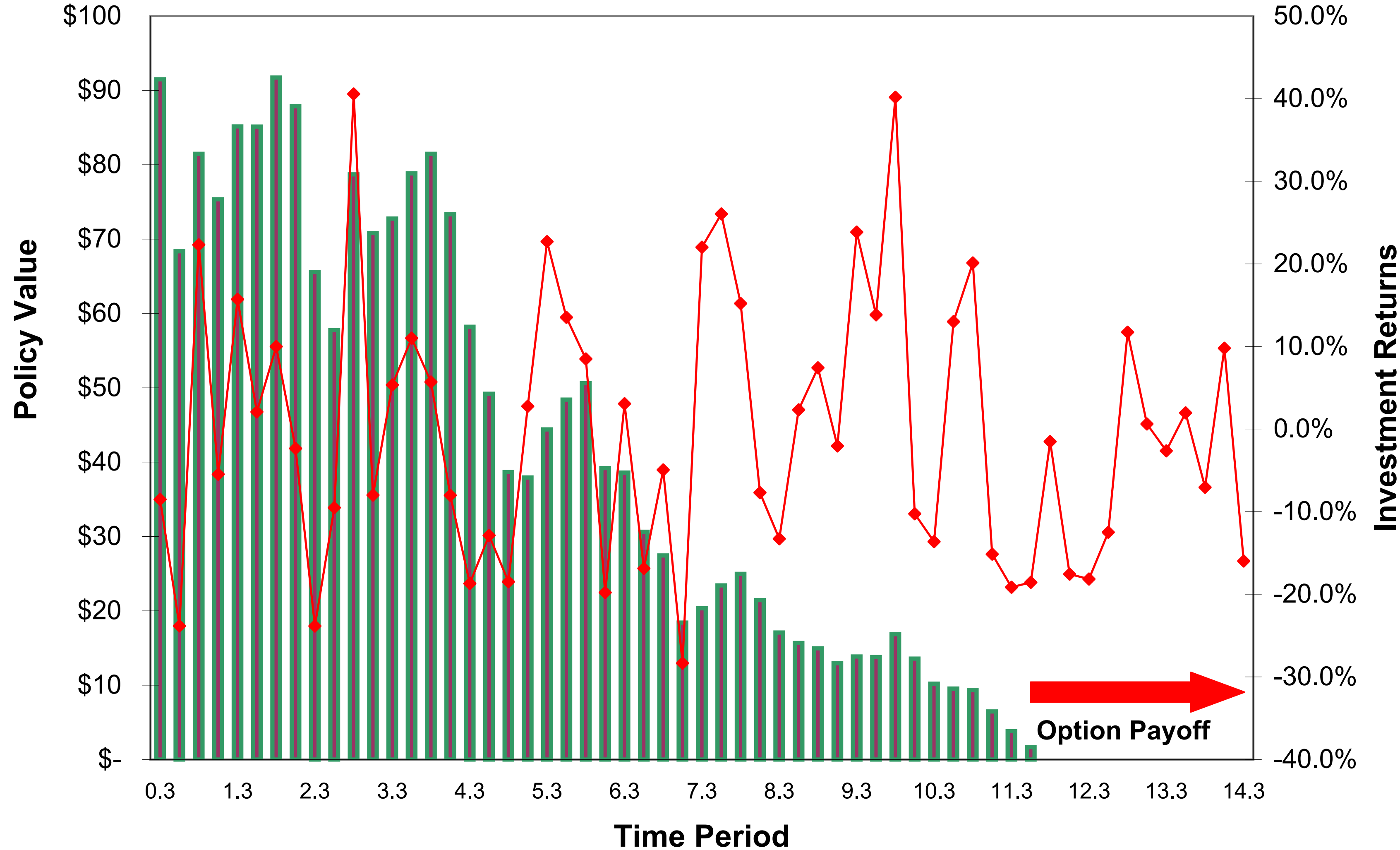


Figure #2: Example of Policy Value under 7% withdrawal and Investment Returns



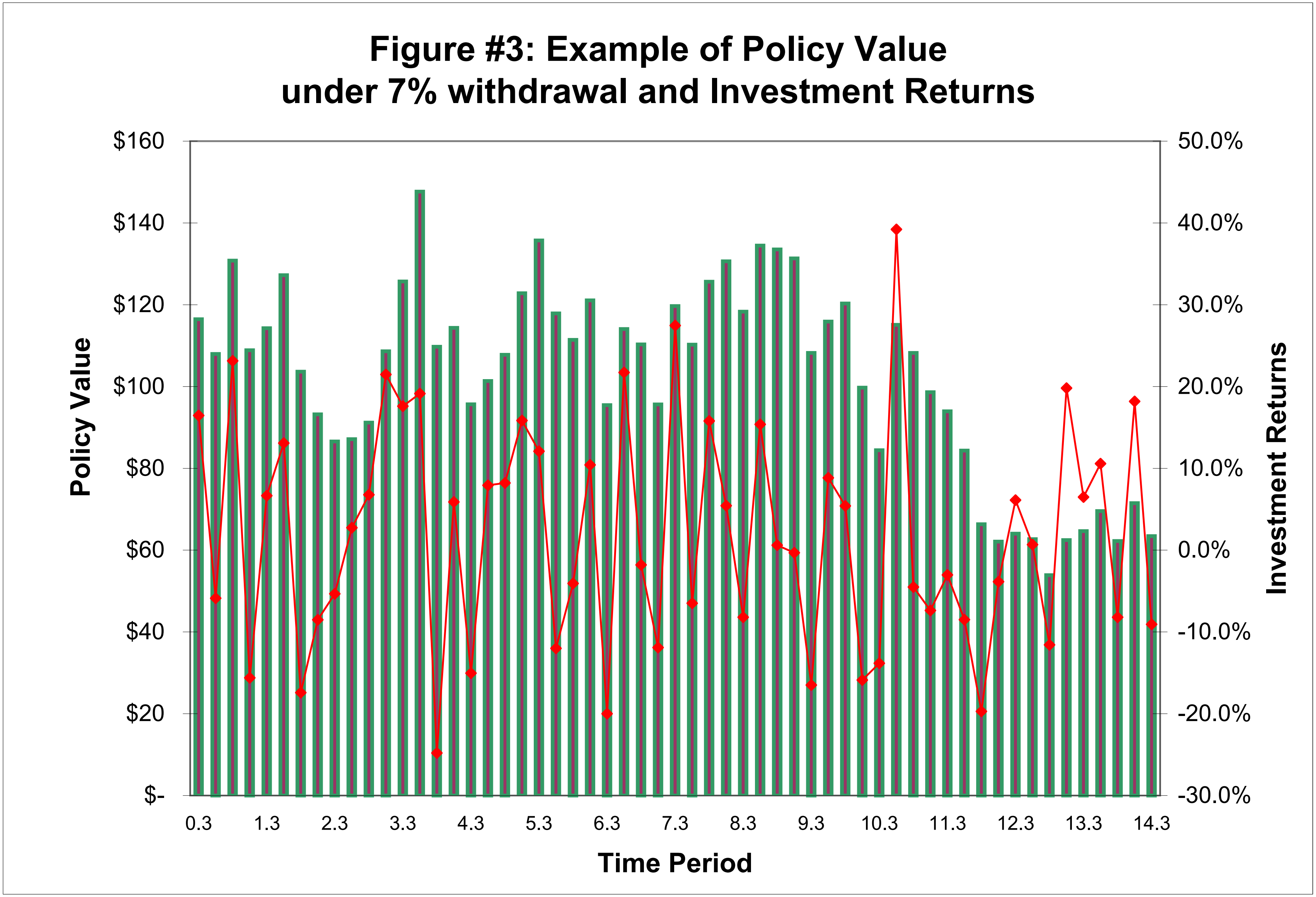


Figure #3: Example of Policy Value
under 7% withdrawal and Investment Returns
Policy Value
Investment Returns
Time Period
$160
$140
$120
$100
$80
$60
$40
$20
$-
50.0%
40.0%
30.0%
20.0%
10.0%
0.0%
-10.0%
-20.0%
-30.0%
0.3
1.3
2.3
3.3
4.3
5.3
6.3
7.3
8.3
9.3
10.3
11.3
12.3
13.3
14.3